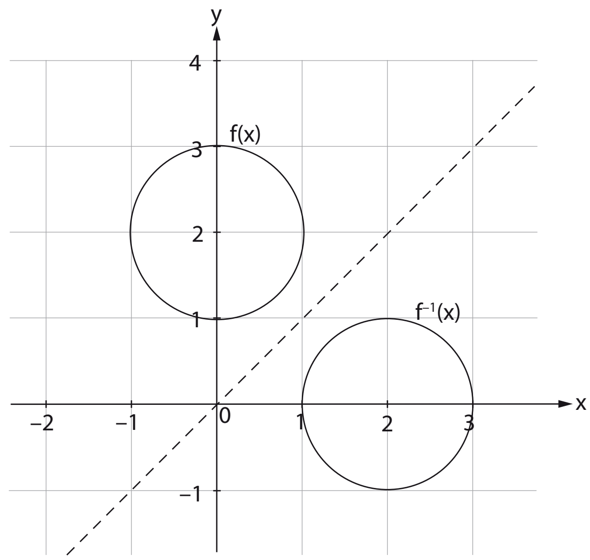
LESSON 4

THE INVERSE GRAPH

The reflection of a graph in the line $y = x$ will be the graph of its inverse.



The line $y = x$ is drawn as the dotted line.

Imagine folding the page along the dotted line, the two circles above would coincide.

The following table shows corresponding points on the two circles.

Circle centre (0;2)	Circle centre (2;0)
(0;1)	(1;0)
(0;3)	(3;0)
(1;2)	(2;1)
(-1;2)	(2;-1)

Notice that every time, the x and y values at the point are simply interchanged.

Finding the equation of the inverse

Step 1: Swap the x and y

Step 2: Make y the subject of the formula, if required.

In the above example, the equation of the circle centre on the y-axis is given by

$$x^2 + (y - 2)^2 = 1.$$

To find the equation of the inverse graph of this circle:

- swap x and y: $y^2 + (x - 2)^2 = 1$
- It is not necessary to make the subject y for the circle equation, but for all functions we do.

What is a "function"?

A function is a relation(graph) where **each x is linked to only one y value.**

The circles above are not functions. Look at the circle with centre on the y-axis.

When $x = 0$, $y = 1$ or $y = 3$. Thus with $x = 0$, there are two values of y linked to it.

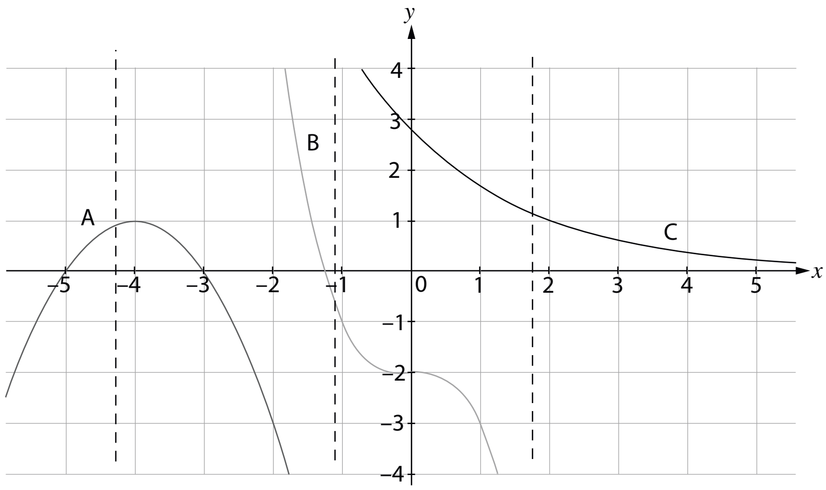
A quick and easy way to check whether a graph is a function or not, is to do the vertical line test.

Vertical Line Test:

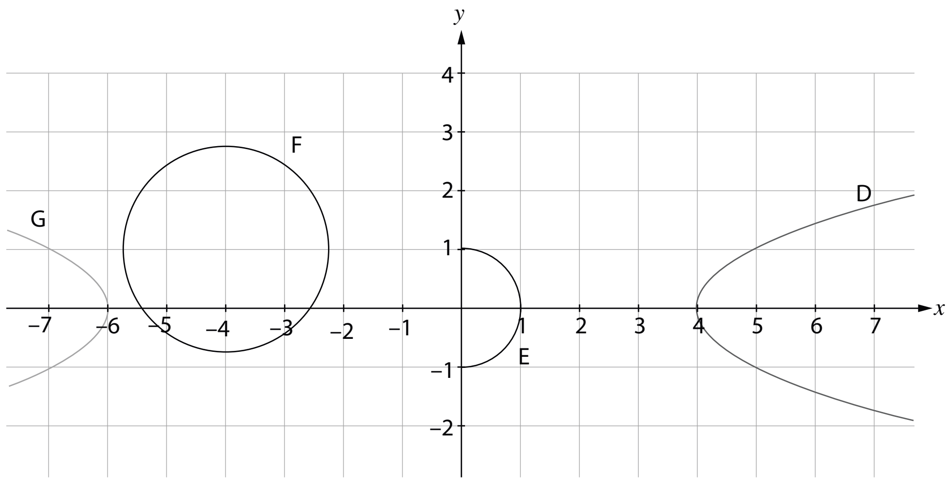
1. Draw a vertical (imaginary) line through the graph.
2. If all vertical lines cut the graph only once, the graph is a function.

In the circles above, there exist many vertical imaginary lines that cut the circle in 2 points. Therefore the circle is not a function.

The graphs labeled A,B and C are all functions. (since vertical lines only cut the graph in one point).



The graphs labeled D,E, F and G below are not functions (since any vertical line can cut the graphs at more than one point).



Notice how the vertical lines cut the graphs in more than one point.

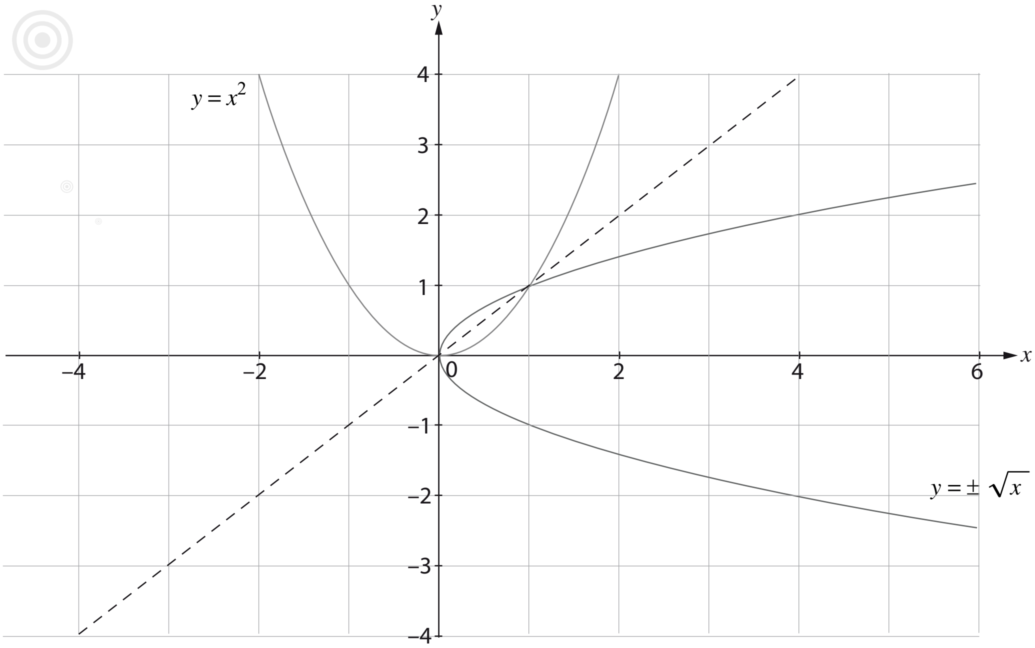
What is an "INVERSE FUNCTION"?

An inverse function is an inverse that is also a function.

Example 1

The graph of $y = x^2$ and its inverse $y = \pm\sqrt{x}$.

Example



To find the equation of the inverse of $y = x^2$,

Swap the x and y: $x = y^2$

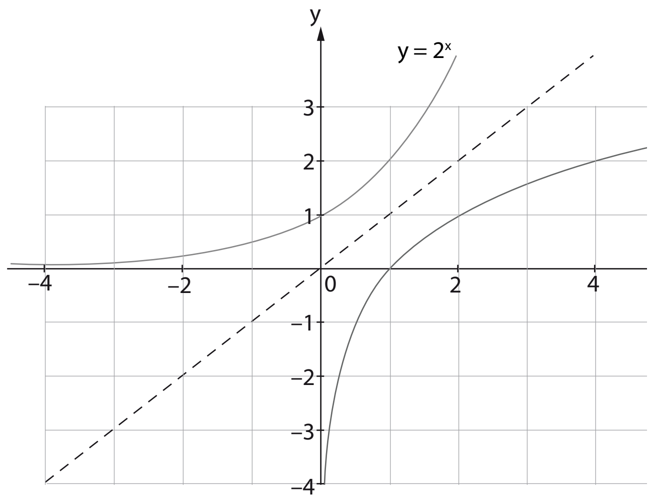
Make y the subject of the formula: $y = \pm\sqrt{x}$

The inverse of $y = x^2$ is not a function.

Example

Example 2

The graph of $y = 2^x$ and its inverse function $y = \log_2 x$:



To find the equation of the inverse of $y = 2^x$,

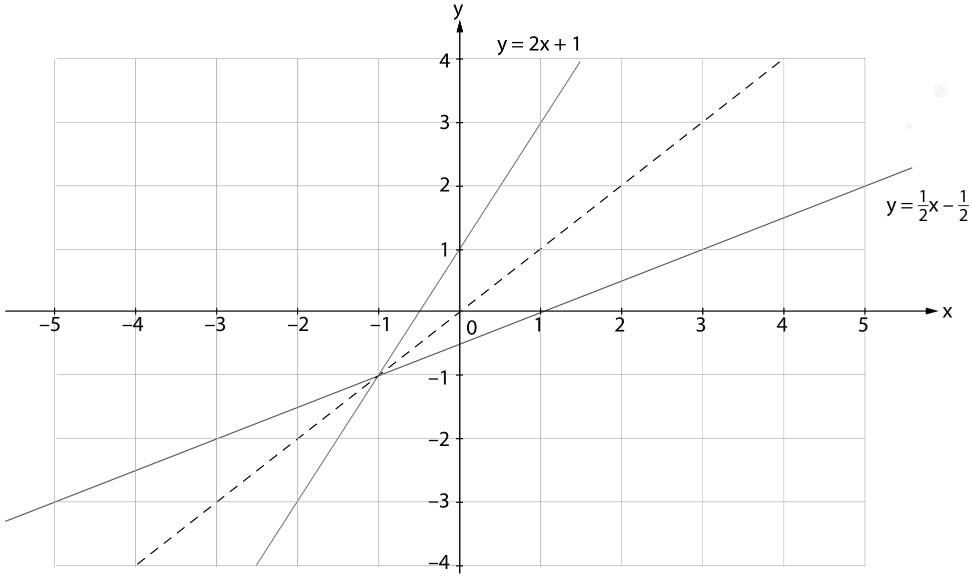
swap the x and y: $x = 2^y$

make y the subject of the formula: $y = \log_2 x$

The graph of $y = 2^x$ has an inverse function.

Example 3

The graph of $y = 2x + 1$ is sketched, together with its inverse.



To find the equation of the inverse:

- Swap the x and the y: $x = 2y + 1$
- Make y the subject: $x - 1 = 2y$

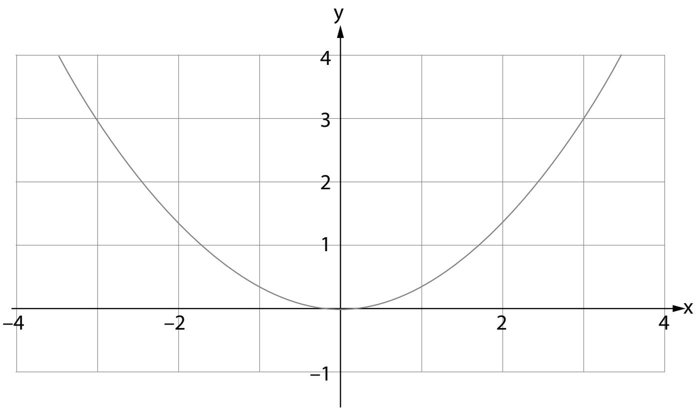
$$\therefore y = \frac{x}{2} - \frac{1}{2}$$

Any straight line graph will have an inverse that is also a function, i.e. $y = mx + c$ will have an inverse function.

Activity

1. For each of the graphs sketched below, state whether the inverse graph will be a function or not. Give reasons.

1.1



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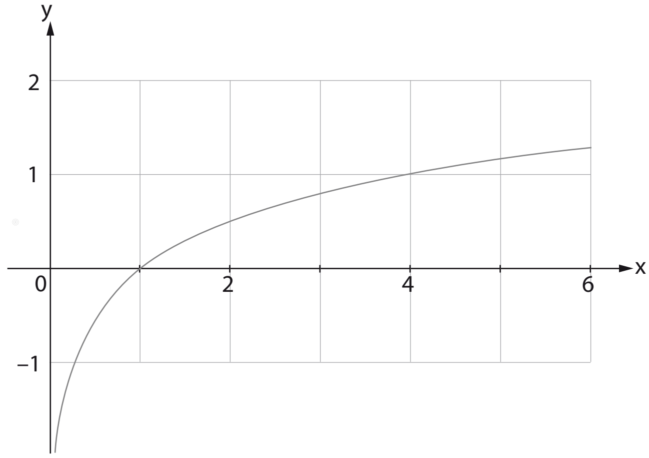
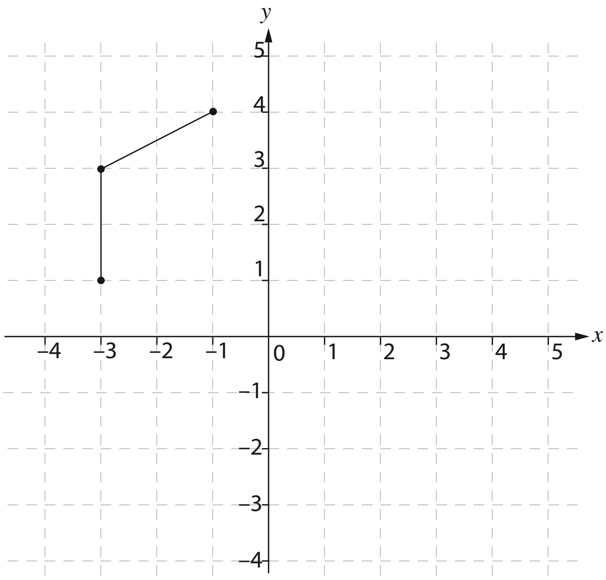
1.2

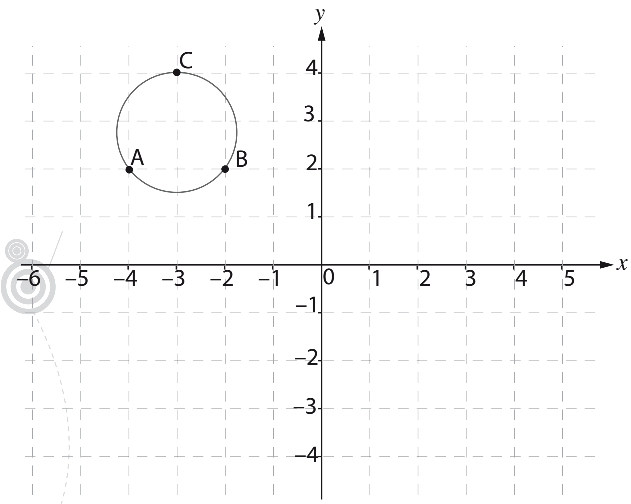


2. For each of the following graphs, draw the inverse graph on the same set of axes.

2.1



2.2



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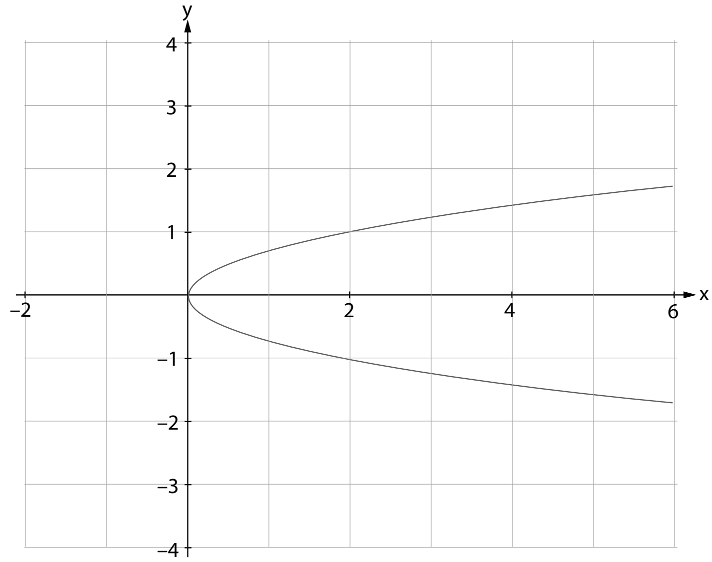
2.3



3. Given: $f(x) = 2x - 3$

3.1 Find the point(s) of intersection of f and f^{-1}, the inverse of f.

3.2 On the same set of axes sketch the graphs of f and f^{-1}

3.3 Determine the equation of f^{-1} in the form $f^{-1}: x \rightarrow \ldots$

4. Given: The graph of $g(x) = a^x$, where a > 0, passes through the point $\left(-1; \frac{2}{3}\right)$.

4.1 Determine the value of a.

4.2 Determine the equation of g^{-1}, the inverse of g, in the form $y = \ldots$

4.3 Sketch the graphs of g and g^{-1} on the same set of axes.

5. Given: $f(x) = -\frac{1}{2}x^2;\ x \geq 0$

5.1 Find the point(s) of intersection of f and f^{-1}, the inverse of f.

5.2 Sketch the graphs of f and f^{-1} on the same set of axes

5.3 Determine the equation of f^{-1} in the form $f^{-1}(x) = \ldots$

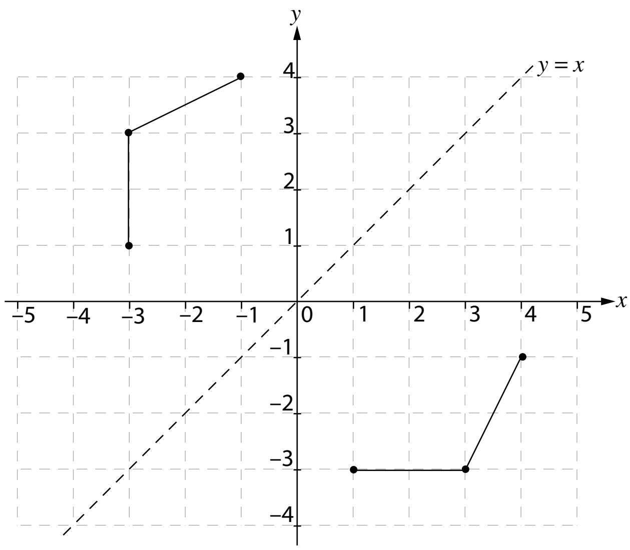
Solutions

1.1 Inverse is not a function. Consider the points (3; 3) and (–3; 3) on the graph. The corresponding points on the inverse will be (3; 3) and (3; –3). Thus for $x = 3$ there is more than one y value.

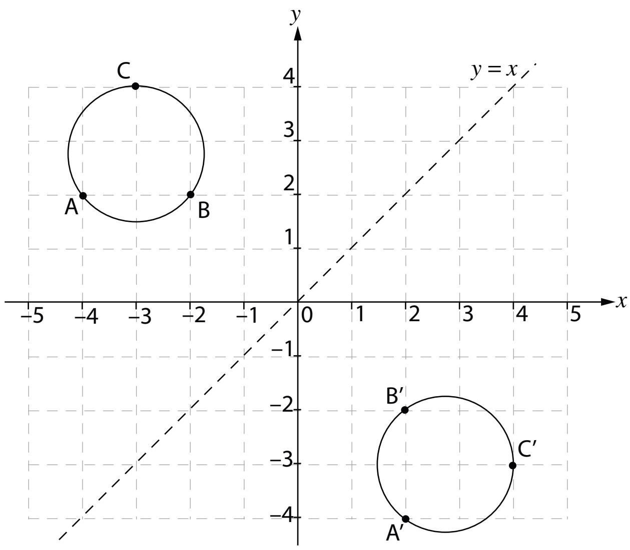
1.2 Inverse is a function. The graph and the inverse have one y-value for each x-value. This is also known as a one to one function. Therefore the inverse will be one to one. If a graph is one to one then its inverse will also be one to one.

1.3 Yes. The graph is one to one. Therefore the inverse will be one to one.

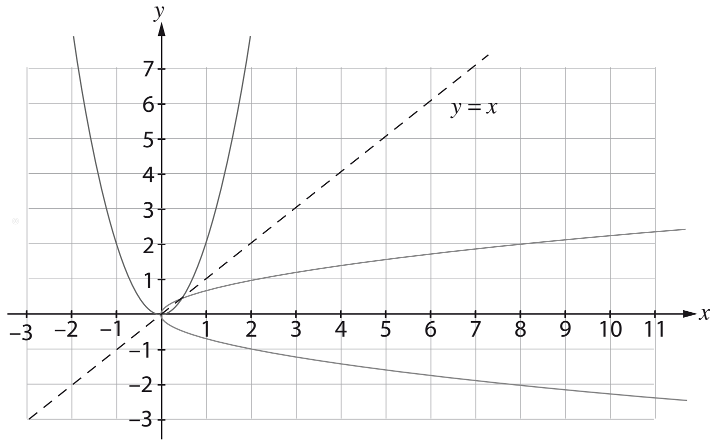
2.1



2.2



2.3



3.1 $2x - 3 = x$

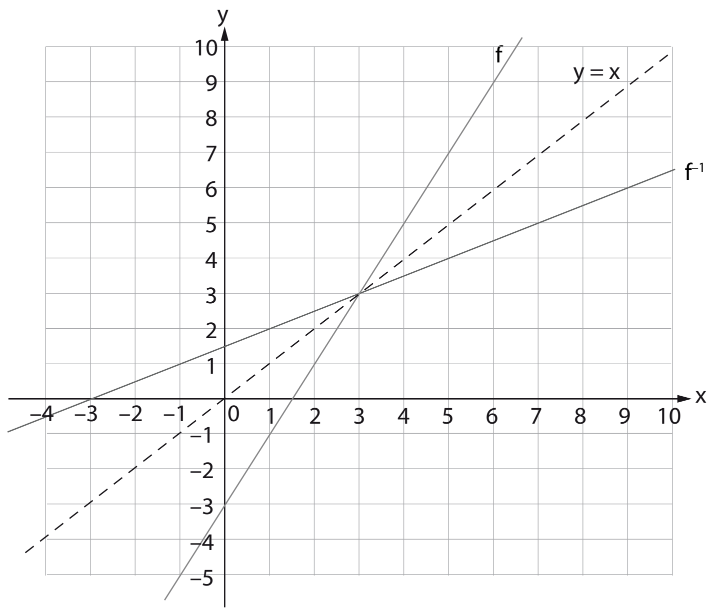
$\therefore\ x = 3$

$\therefore y = 2(3) - 3$

$\therefore$ point of intersection (3;3)

If f is a one-to-one function, then the point of intersection between f and f^{-1} is always along the line $y = x$ the x- and y-co-ordinate values are always the same.

3.2



3.3 For inverse,

$x = 2y + 3$ (because f is a one-to-one function)

$\therefore\ 2y = x - 3$

$\therefore\ y = \frac{1}{2}x - \frac{3}{2}$

$\therefore\ f^{-1}: x \rightarrow \frac{1}{2}x - \frac{3}{2}$

4.1 $y = a^x$

$\therefore\ \frac{2}{3} = a^{-1}$

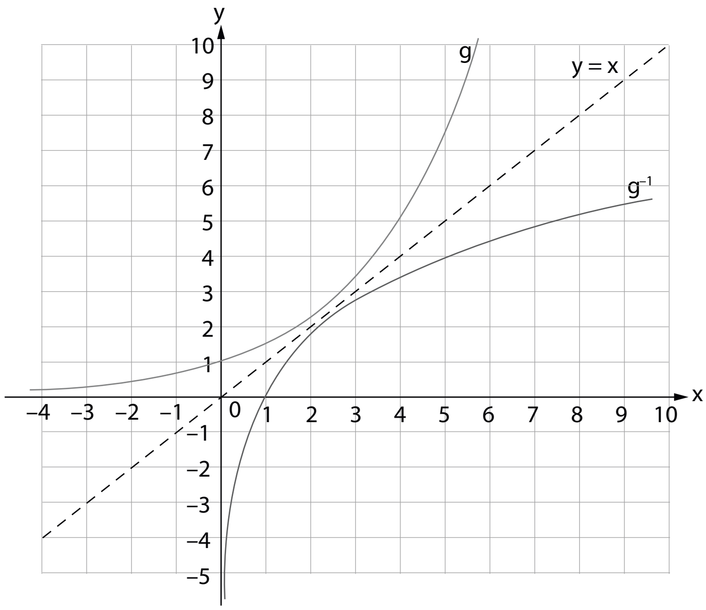
$\therefore\ a = \frac{3}{2}$

4.2 For the inverse,

$x = \left(\frac{3}{2}\right)^y$

$y = \log_{\frac{3}{2}} x$

4.3



5.1 $-\frac{1}{2}x^2 = x$ (because f is a one-to-one function)

$\therefore -x^2 = 2x$

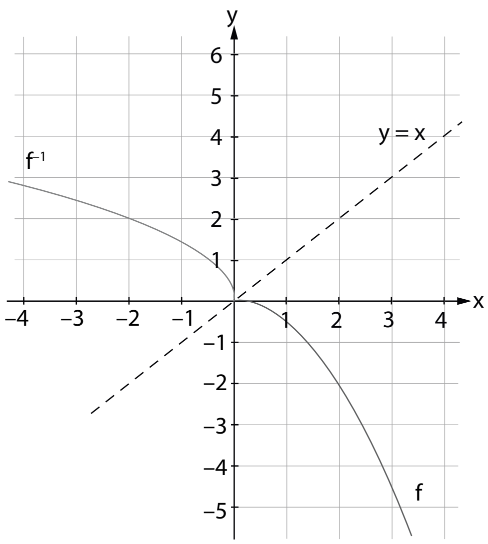
$\therefore x^2 + 2x = 0$

$\therefore x(x+2) = 0$

$\therefore x = 0$ or $x = -2$(N.A)

Therefore, the point(s) of intersection are (0; 0)

5.2



5.3 For the inverse: (swap x and y)

$x = -\frac{1}{2}y^2;$

$\therefore 2x = -y^2;$

$\therefore y^2 = -2x;$

$y = \pm\sqrt{-2x};$

$\therefore f^{-1}(x) = \pm\sqrt{-2x};\ x \leq 0$ (Note: $x \leq 0$, since we cannot square root a negative.)

Note: The restriction $x \leq 0$ must be included because f^{-1} is the inverse of a restriction on f.