FINANCIAL MATHEMATICS

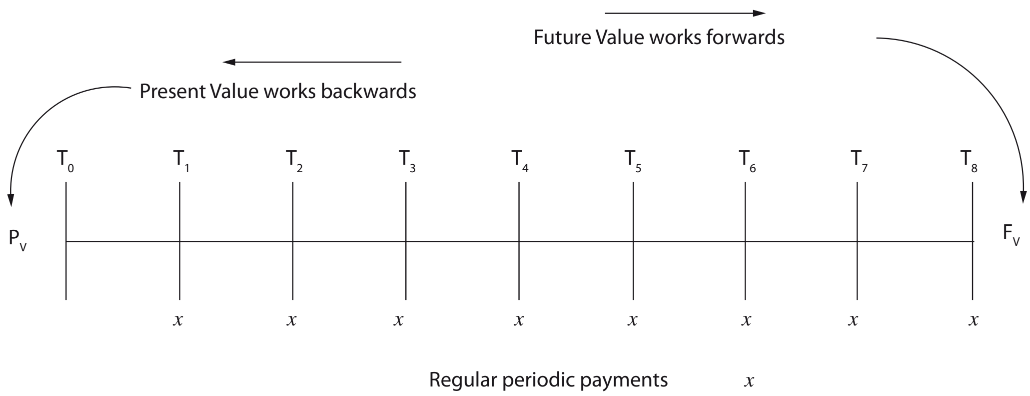
Annuities

What is an annuity?

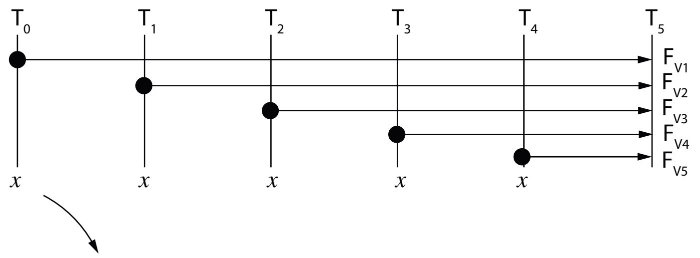
The term ***annuity*** is used in financial mathematics to refer to any terminating sequence of regular fixed payments over a specified period of time. Loans are usually paid off by an annuity. If payments are not at regular (irregular) periods, we are not working with an annuity. We get two types of annuities:

The ordinary annuity	The annuity due
This is an annuity whose payments are made at the end of each period. (At the end of each week, month, half year, year, etc.) Paying back a car loan, a home loan…	This is an annuity whose payments are made at the beginning of each period. Deposits in savings, rent payments, and insurance premiums are examples of annuities due.

On the time line:
If we look at the timeline, it is clear to see that if we are looking at investing money into an account, then we will be working with the **future value** of these payments. This is so because we are saving up money for some use one day in the future. If we want to consider the **present value** of a series of payments, then we will be looking at a scenario where a loan is being paid off. This is so because we get the money today, and pay that money with interest back to the financing company some time in the future.



Future value of an annuity:
When we calculate the future value of an annuity, it is important to realize that each of the regular payments is a present value that will collect interest during the term / period of the investment. The present values then become a sequence of future values when we move them forward towards the end of the time line. Collectively, their sum results into a future value of the investment. The future value of the annuity thus consists of the sum of each payment's future value, and forms a geometric sequence of which we determine the sum of the payments.



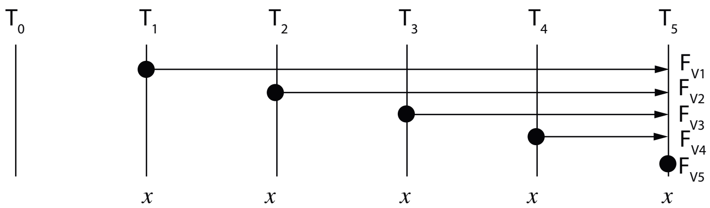
The x values are all present values which need to go forward to become future values when the annuity matures. This 'going forward' occurs by adding interest for the period.

Lets us see how this works:

In the diagram alongside we see that 5 payments have been made into an account at regular yearly intervals. Each payment into this account happened at a different time, but they are all spaced at once a year. Each payment must therefore move to timeline T_5 to become a future value. That means that when we paid in the Rx at the beginning of each year, that Rx was a present value at the time which became a future value as it moved forwards on the timeline. The sum of all these separate future values gives us the future value of the annuity. Note that the above scenario was an annuity due as each payment was made at the beginning of the period.

The formula:
Let us invest Rx monthly into a saving account that pays *r* % interest per annum compounded annually, for a total of *n* payments. The first payment is made at the end of the month and the last payment at the end of *n* months. So this is an **ordinary annuity**.

The timeline:



At T_1:	We pay in Rx at the end of the first year
At T_2:	This Rx has accrued one year's interest to become a future value at compound interest. So we get that here using the compound increase formula $F_V = P_v(1 + i)^n$: F_V at $T_2 = x(1 + i)^1$. At the end of the second month we add another payment of Rx. So now the money in the bank has increased by adding interest to the first payment and then adding another Rx to the account to become F_v at $T_2 = x(1 + i)^1 + x$.
At T_3:	The moneys accumulated at T_2 now moves forward to T_3 by adding interest to the whole amount: $F_V =$ at $T_3 = [x(1 + i)^1 + x](1 + i)$ $= x(1 + i)^2 + x(1 + i)$ We finally add another payment so that this amount becomes F_V at $T_3 = x(1 + i)^2 + x(1 + i) + x$. Note that the earlier two payments have interest that is accumulating as it moves forward.

At T_4:	The money from T_3 moves forward one period and we then add the new payment to get: F_V at $T_4 = [x(1+i)^2 + x(1+i) + x](1+i) + x$ $= x(1+i)^3 + x(1+i)^2 + x(1+i) + x$
At T_5:	The money from T_4 moves forward one period and we then add the new payment to get: F_V at $T_5 = [x(1+i)^3 + x(1+i)^2 + x(1+i) + x](1+i) + x$ $= x(1+i)^4 + x(1+i)^3 + x(1+i)^2 + x(1+i) + x$

If we want to do this for 120 payments the process will be very lengthy, so we need a shorter method that does the same thing. If we look at the sequence of terms we see that:

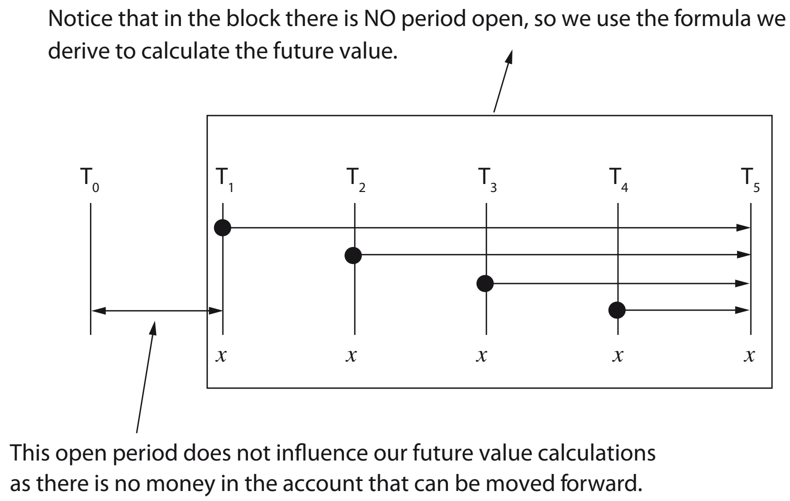
$$F_V = x + x(1+i) + x(1+i)^2 + x(1+i)^3 + x(1+i)^4$$

This is a geometric sequence of values with first term x and constant ratio $(1 + i)$.

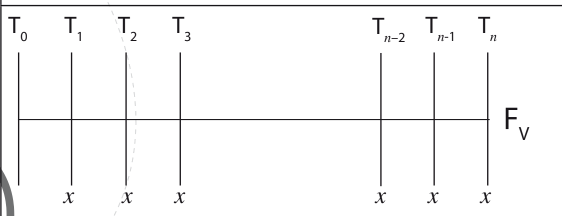
Let's see how this works:

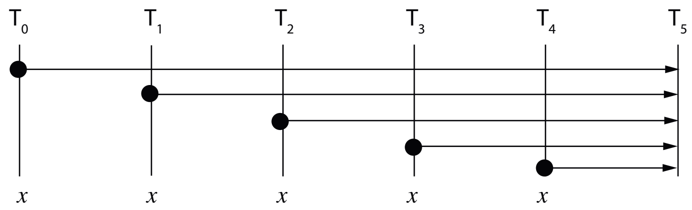
$$S_n = \frac{a(r^n - 1)}{r - 1}: a = x, r = (1+i) \text{ and } n = 5$$
$$\therefore S_5 = \frac{x((1+i)^5 - 1)}{(1+i) - 1} = x\left[\frac{(1+i)^5 - 1}{i}\right]$$

So the future value of these five payments will be: $F_V = x\left[\frac{(1+i)^5 - 1}{i}\right]$.



There are two important things to mention here. We are moving forward on the timeline so the powers of the $(1 + i)$ ratio will be positive as we are adding interest to every payment. Secondly there is no period open with payments.

So if we wish to generalize this formula, we can say that for the following ordinary annuity for n periodic payments into an account that pays an effective r% per period, the timeline will look as follows:	
	And the future value will be: $F_V = x\left[\frac{(1+i)^n - 1}{i}\right]$



If we work with an **annuity due** where the first payment is made immediately and the last payment is made at the beginning of the last period, then the situation looks rather different since the period at the end influences the future value calculation. The scenario changes to the following:

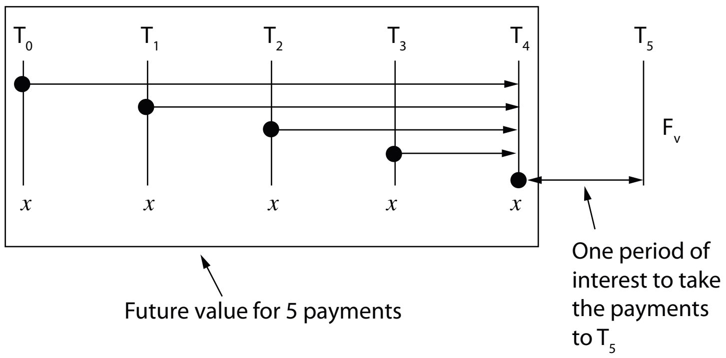
At T_1:	F_V at $T_1 = x(1+i)^1 + x$
At T_2:	F_V at $T_2 = x(1+i)^2 + x(1+i) + x$
At T_3:	F_V at $T_3 = x(1+i)^3 + x(1+i)^2 + x(1+i) + x$
At T_4:	F_V at $T_4 = x(1+i)^4 + x(1+i)^3 + x(1+i)^2 + x(1+i) + x$
At T_5:	F_V at $T_5 = x(1+i)^5 + x(1+i)^4 + x(1+i)^3 + x(1+i)^2 + x(1+i)$

If we look at the sequence of terms we see that:

$$F_V = x(1+i) + x(1+i)^2 + x(1+i)^3 + x(1+i)^4 + x(1+i)^5$$

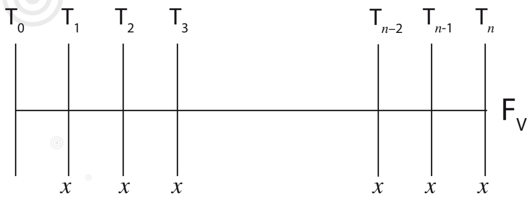
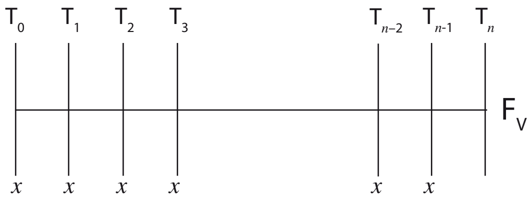
We can remove a common factor of $(1 + i)$ to get:

$$F_V = (1+i)\left[x + x(1+i) + x(1+i)^2 + x(1+i)^3 + x(1+i)^4\right]$$



Now we already know that for the square brackets we have a formula which says that $S_5 = x\left[\frac{(1+i)^5 - 1}{i}\right]$. So we use this to get $F_V = x\left[\frac{(1+i)^5 - 1}{i}\right](1 + i)$. This would be the same as moving all the payments to a future value at T_4, starting at T_0, and then adding one more period of interest to take this accumulated future value to the end of the five year period. This is very important to remember, because this is exactly how the future value formula works. If you understand fully how to adjust the standard formula, then you won't have major difficulties with the future value formulae for finance. Let us summarize the future value of an annuity by looking at different timelines and showing how the standard formula adjusts to accommodate the different payment periods:

For the ordinary annuity for *n* periodic payments into an account that pays an effective *r*% per period,

the timeline will look as follows: 	The formula: $F_V = x\left[\frac{(1+i)^n - 1}{i}\right]$
For the annuity due for *n* periodic payments into an account that pays an effective *r*% per period,	
the timeline will look as follows: 	The formula: $F_V = x\left[\frac{(1+i)^n - 1}{i}\right](1+i)$

Some worked examples:

Example

Example 1:

An investment of R300 per month, with the first, of 30 payments, made in one month's time, matures to Rx after three years. Interest is paid at a rate of 18% per annum, compounded monthly. Determine x.

Effective monthly interest rate:

$\frac{0,18}{12}$ or $\frac{18}{1\,200} = 0,015.$

So:

$F_V = x\left[\frac{(1+i)^n - 1}{i}\right]$

$\therefore\ F_V = 300\left(\frac{(1,015)^{36} - 1}{0,015}\right)$

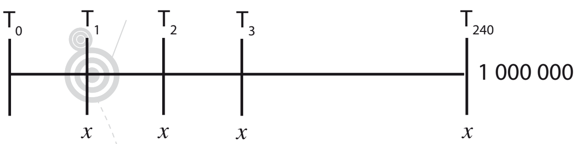
$\therefore\ F_V = 300(47,27596921)$

$\therefore\ F_V = \text{R}14\ 182,79$

Example

Example 2

How much money must be invested monthly into an ordinary annuity to realise R1 000 000 in 20 year's time if the current rate of investment is calculated at an effective 9% per annum compounded annually? (The payments stretch over the 20 year period)



We need an effective monthly rate:

$\left(1 + \frac{i_{12}}{12}\right)^{12} = (1+i)$

$\therefore\ \frac{i_{12}}{12} = \sqrt[12]{1,09} - 1 = 0,00720732331$

$F_V = x\left(\frac{(1+i)^n - 1}{i}\right)$

$\therefore\ 1\ 000\ 000 = x\left(\frac{(1,00720732331)^{240} - 1}{0,00720732331}\right)$

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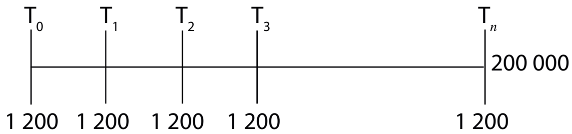
$\therefore 1\ 000\ 000 = x(638{,}8517021)$

$\therefore F_V = \text{R}1\ 565{,}31$

Example 3

Example

How long should an investor continue to make monthly investments of R1 200 at a rate of 12% per annum compounded monthly if he wishes to have at least R200 000 in order to buy a car cash? Assume that his first payment is immediately and that his last payment is made on the day the investment matures.



Effective monthly interest rate:

$\frac{0{,}12}{12}$ or $\frac{12}{1\ 200} = 0{,}01.$

Notice that the payments start immediately and end on the last day. So the timeline shows payments from T_0 to T_n which is $n + 1$ payments:

$$F_V = x\left[\frac{(1+i)^{n+1} - 1}{i}\right]$$

$$\therefore 200\ 000 = 1\ 200\left(\frac{(1{,}01)^{n+1} - 1}{0{,}01}\right)$$

$$\therefore \left(\frac{200\ 000}{1\ 200}\cdot 0{,}01\right) + 1 = (1{,}01)^{n+1}$$

$$\therefore \frac{8}{3} = (1{,}01)^{n+1}$$

$$\therefore n + 1 = \frac{\log \frac{8}{3}}{\log 1{,}01} = 98{,}5725\ldots$$

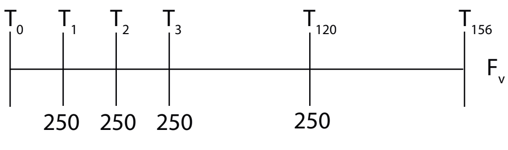
$$\therefore n = 97{,}5725$$

$$\therefore n = 98 \text{ months}$$

Example 4

Example

Manuel decides to save a monthly amount of R250 for the next ten years. His bank offers him an interest rate of 8% p.a. compounded monthly for this period. After the ten years the bank will offer him 12% p.a. compounded monthly if he does not withdraw the money for the following three years. How much will he have in the bank at the end of the 13 years?



Effective monthly interest rate:

$i_1 = \frac{0{,}08}{12}$ or $\frac{8}{1\ 200} = 0{,}00\dot{6}$ and $i_2 = \frac{0{,}12}{12}$ or $\frac{12}{1\ 200} = 0{,}01.$

Notice that the payments stop after ten years and the money then remains in the account to accumulate interest for a further three years:

$$F_V = x\left(\frac{(1+i_1)^{120} - 1}{i}\right)(1 + i_2)^{36}$$

$$\therefore F_V = 250\left[\frac{(1 + \frac{0{,}08}{12})^{120} - 1}{\frac{0{,}08}{12}}\right](1{,}01)^{36}$$

$$\therefore F_V = 250(182{,}9460352)(1{,}01)^{36}$$

$$\therefore F_V = 45736{,}5088(1{,}01)^{36}$$

$\therefore F_V = \text{R}65\ 438,37$

Notice that the annuity matures the moment the payments end, and then this value simply accumulated more interest at a normal compound interest rate.

Activity

Activity 1

1. Peter invests R300 per month into an account that pays 14% p.a. compounded monthly for a period of five years. His first payment is immediately, and his last payment is on the day the investment matures. How much does he have in this account at the end of 5 years?

2. Bongiwe wants to save up some money so that she can give her unborn daughter R16 000 on her 16^{th} birthday. On the day that her daughter is born, she starts making equal monthly payments into an account that pays 8% p.a. compounded monthly. Her last payment into this account is due one month before her daughter turns 16.

2.1 Calculate the size of the monthly payment.

2.2 How much should Bongiwe invest monthly if she wants to give her daughter R21 000 on her 21st birthday, instead of the R16 000 on her 16th birthday? Her last payment into this account is due one month before her daugher turns 21.

3. Which investment will be the better option: monthly deposits of R100 into an account paying 12%p.a. compounded monthly or quarterly investments of R300 into an account paying 4% per quarter. Both investments run for one year.

4. Sam makes monthly deposits of R200 into an ordinary annuity for a period of four years and earns interest at 6%p.a. compounded monthly. At the end of each year, he deposits an additional R1 000 into this account. How much money will Sam have in this account at the end of four years?

Sinking Funds

What is a sinking fund?

A **sinking fund** is an investment that is made to replace expensive equipment / items in a few years' time. It is used as a "savings account" that will accumulate funds over a period of time, which will enable the investor to purchase expensive items or to fund expensive capital outlays in a few years' time.

The workings of the sinking fund problems are best explained by looking at a few worked problems.

Example

Example 1

Machinery is purchased at a cost of R550 000 and is expected to rise in cost at 15% per annum, compound interest, and depreciate in value at a rate of 8% per annum compounded annually.

A sinking fund is started to make provision for replacing the old machine. The sinking fund pays 16% per annum compounded monthly, and you make monthly payments into this account for 10 years, starting immediately and ending one month before the purchase of the new machine. Determine:

1. the replacement cost of a new machine ten years from now
2. the scrap value of the machine in ten years time.
3. the monthly payment into the sinking fund that will make provision for the replacement of the new machine.

Solution

Solution

We need to separate the information that is given so that we do not mix up some rates and some periods. So:

Current Machine :

P_V = R550 000

Appreciation rate: r = 15%p.a. compounded annually

Depreciation rate: r = 8% p.a. compounded annually

Term: $n = 10$ years

Sinking Fund :

Interest rate: r = 16% p.a. compounded annually

Payment = Rx per month

Term: $n = 10$ years

1. Since the cost of a new machine appreciates at an effective yearly rate of 15% p.a :

$$F_V = P(1 + i)^n$$

$$\therefore F_V = 550\,000(1{,}15)^{10}$$

$$\therefore F_V = \text{R}2\,225\,056{,}76$$

This future value is known as the REPLACEMENT COST.

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2. Since the old machine depreciates at an effective yearly rate of 8% p.a :

$F_V = P(1 + i)^n$

$\therefore F_V = 550\ 000(1 - 0{,}08)^{10}$

$\therefore F_V = 550\ 000(0{,}92)^{10}$

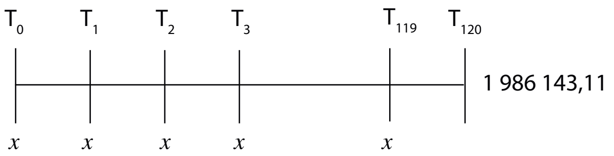
$\therefore F_V = R238\ 913{,}65$

This depreciated value is also called the SCRAP VALUE.

3. The scrap value is always used as a part payment on the new machine. So if we take the replacement cost and we subtract the scrap value, we obtain the value of the sinking fund.

So: Sinking fund = R2 225 056,76 – R238 913,65 = R1 986 143,11

The Timeline:



The effective monthly rate: $\frac{0{,}16}{12}$ or $\frac{16}{1200} = 0{,}01\dot{3}$

$F_V = x\left(\frac{(1 + i)^n - 1}{i}\right)(1 + i)$

$\therefore 1986143{,}11 = x\left[\frac{(1 + \frac{0{,}16}{12})^{120} - 1}{\frac{0{,}16}{12}}\right](1 + \frac{0{,}16}{12})$

$\therefore 1986143{,}11 = x(296{,}4715095)$

$\therefore x = R6\ 699{,}27$

It is clear to see that there are a few steps that will be standard in each problem that involves a sinking fund. Firstly we need to always calculate the replacement cost, and the scrap value of the machine. To find the value of the sinking fund, we need to determine the difference between the replacement value and the scrap value as the old machine is always sold to defray costs of purchasing a new machine. We then set up a future value annuity to determine the amount of the monthly instalments.

Example 2

A compact disc press is purchased for R 1,2 million and is expected to rise in cost at a rate of 9 % per annum compounded annually, whilst it will depreciate at a rate of 7,5% per annum compounded annually. A sinking fund is set up to make provision for the replacement of the machine in ten years' time, and pays interest at a rate of 9,2% per annum compounded monthly.

1. Determine the monthly amount that has to be invested into the sinking fund to realize enough money for a replacement machine in ten years' time. Payments start immediately and end on the day that the replacement machine is purchased.

2. After five years new technology in Compact Discs are introduced to the market. This machine will cost R 2 million. If you decide to replace your current machine immediately, how much money will you have to borrow to purchase the new equipment, if you use the sinking fund and the sales of the old machine toward paying for this new machine?

Solution

Solution

Current Machine :

P_V = R1 200 000

Appreciation rate: r = 9% p.a.c.a

Depreciation rate: r = 7,5% p.a.c.a

Term: $n = 10$ years

1. Replacement cost:

   $F_V = P_V(1 + i)^n$

   $\therefore F_V = 1\ 200\ 000(1{,}09)^{10}$

   $\therefore F_V = R2\ 840\ 836{,}41$

   Scrap Value:

   $F_V = P_V(1 - i)^n$

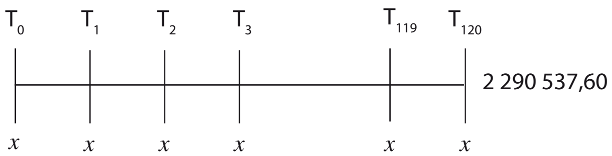
   $\therefore F_V = 1\ 200\ 000(1 - i)^n$

   $\therefore F_V = R550\ 298{,}81$

   So the value of the sinking fund:

   R2 840 836,41 – R550 298,81 = R2 290 537,60

   The timeline:



   Notice that there are 121 payments this time round.

   So:

   The effective monthly rate: $\frac{0{,}092}{12}$ or $\frac{9{,}2}{1200} = 0{,}007\dot{6}$

   $F_V = x\left(\frac{(1 + i)^n - 1}{i}\right)$

   $\therefore 2290537{,}60 = x\left(\frac{(1{,}007\dot{6})^{121} - 1}{0{,}007\dot{6}}\right)$

   $\therefore 2290537{,}60 = x(198{,}21807)$

   $\therefore x = R11\ 555{,}64$

2. We now need to move everything to the five year point instead of the ten year point.

   Scrap Value:

   $F_V = P_V(1 - i)^n$

   $\therefore F_V = 1\ 200\ 000(1 - 0{,}075)^5$

   $\therefore F_V = R812\ 624{,}50$

   The sinking fund at the end of five years:

   $F_V = x\left(\frac{(1 + i)^n - 1}{i}\right)$

   $\therefore F_V = 11\ 555{,}64\left(\frac{(1{,}007\dot{6})^{61} - 1}{0{,}007\dot{6}}\right)$

   $\therefore F_V = 11\ 555{,}64(77{,}40269292)$

   $\therefore F_V = R894\ 437{,}65$

   So we need to borrow R2 000 000 – R894 437,65 – R 812 624,50
   = R292 937,85.

Activity

Activity 2

1. A company purchases a bus for a price of R x. It is expected to have a useful life of n years and a scrap value of 18% of its original purchase price. If the annual depreciation rate is estimated to be 21% p.a. on the reducing balance, determine the value of n.

2. A machine which costs R120 000 is estimated to have a useful life of 10 years and then a scrap value of R45 000. Determine the annual depreciation rate on:

2.1 the reducing balance basis.

2.2 on the straight line basis.

3. A hydraulic lifter costs R550 000 and is expected to have a useful lifetime of 8 years. It depreciates at 10% p.a. on the reducing balance basis. The cost of a replacement lifter is expected to escalate at 18% p.a. effective. A sinking fund is set up to finance the replacement hydraulic lifter in 8 years' time. Find, at the time of purchase of the new hydraulic lifter:

3.1 The scrap value of the old hydraulic lifter.

3.2 The expected cost of a new hydraulic lifter.

3.3 The value that the sinking fund must attain, if the scrap value of the old hydraulic lifter is used to defray expenses.

3.4 The value of the monthly installments that are made into the sinking fund if payments start immediately and end on the day of the replacement and the sinking fund earns interest of 12% p.a. compounded monthly.

4. A printing press currently costs R850 000. The value of the machine is expected to drop at a rate of 7% per annum simple interest, whilst the cost of a replacement machine escalates at a rate of 14% p.a. compounded annually. The press is expected to have a useful lifetime of 8 years.

4.1 Calculate the scrap value of the old machine.

4.2 Calculate the cost of the replacement machine.

4.3 Calculate the amount needed to replace the old machine, if the scrap value is used as part of the payment for the new machine.

4.4 A sinking fund is set up to provide for this balance, paying interest at 15% p.a. compounded monthly. Determine the monthly amount that should be paid into the sinking fund to realize this. Payments start immediately and end 6 months before replacement.

5. Michelle is the proud owner of an interior decorating business and needs to purchase machinery to restore antique furniture. The cost of this equipment is R 1,2 million and it is estimated that the machinery will depreciate in value at a rate of 9% per annum compounded annually. She decides to invest money into a sinking fund, to make provision for the replacement of the equipment in five year's time. The sinking fund pays 14% per annum compounded quarterly. It is estimated that the replacement cost of the new equipment rises at a rate of 6% per annum compounded annually.

5.1 Determine the scrap value of the current equipment in five years' time.

5.2 Determine the replacement value of the new machinery in five years' time.

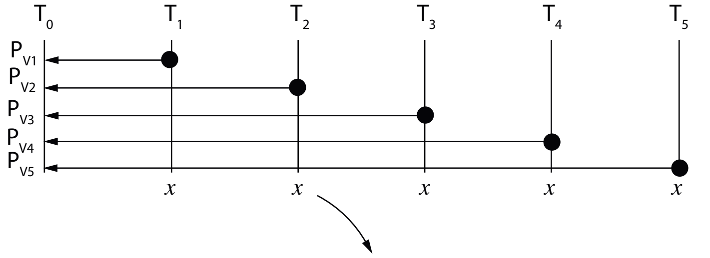
5.3 If the scrap value of the machine is used toward paying for the new machinery, determine the quarterly amount that must be paid into the sinking fund to make provision for the replacement of the old machinery in five years' time. The first payment is made immediately and the last payment at the end of five years.

5.4 Three years after purchasing the current machine, new technology becomes available and Michelle wants to replace the machinery immediately. How much money will she have available to do so if at this point in time she has made 13 payments into the sinking fund?

Present Value Annuities

When we calculate the present value of an annuity, each of the regular payments are **future** (accumulated) values that are made up of **capital and interest added**. These future values then become a sequence of present values when we move them **backwards** towards the beginning of the time line. Collectively, their sum results into a present value of the investment or loan- that is each payment without interest. Each future value (payment) has thus moved backwards on the timeline to become a present (principal) value at the beginning of the time line. The **present value of the annuity** thus consists of the sum of each payment's present value, and forms a geometric sequence of which we determine the sum of the payments.

We say that an interest bearing loan / debt is **amortised** if both the principal and the interest are paid by means of an annuity.

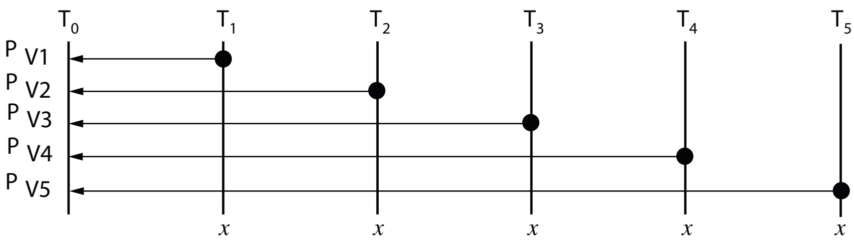


The x values are all future values which need to go backwards to become present values at the beginning of the time line. By going backwards we are removing interest for the period.

Lets us see how this works:

In the diagram above we see that 5 payments have been made into an account at regular yearly intervals. Each payment into this account happened at a different time, but they are all spaced at once a year. Each payment must therefore move to timeline T_0 to become a present value. That means that when we paid in the Rx at the end of each period, Rx was a future value at the time which became the equivalent value as it moved backwards on the timeline. The sum of all these separate present values gives us the present value of the annuity. Note that the above scenario was an ordinary annuity as each payment was made at the end of the period.

The formula:
Let us pay back a loan by making yearly down payments of R x into this account. Interest is charged at r % per annum compounded annually, for a total of n payments. The first payment is made at the end of the month and the last payment at the end of n months. So this is an **ordinary annuity**.
The timeline:



We move each of the payments back to T_0.

So:

Since: $F_V = P_V(1+i)^n$

$\therefore P_V = \frac{F_V}{(1+i)^n}$

$\therefore P_V = F_V(1+i)^{-n}$

From T_1:	$P_{V_1} = x(1+i)^{-1}$	
From T_2:	$P_{V_2} = x(1+i)^{-2}$	
From T_3:	$P_{V_3} = x(1+i)^{-3}$	If we now add all these present values together we get: $P_V = x(1+i)^{-1} + x(1+i)^{-2} + x(1+i)^{-3} + x(1+i)^{-4} + x(1+i)^{-5}$
From T_4:	$P_{V_4} = x(1+i)^{-4}$	
From T_5:	$P_{V_5} = x(1+i)^{-5}$	

So now:

$a = x(1+i)^{-1}$; $r = (1+i)^{-1}$; $n = 5$

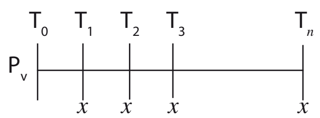
$$\begin{aligned} S_5 &= \frac{x(1+i)^{-1}[1-(1+i)^{-5}]}{1-(1+i)^{-1}} \\ &= \frac{x(1+i)^{-1}[1-(1+i)^{-5}]}{1-(1+i)^{-1}} \cdot \frac{(1+i)^1}{(1+i)^1} \\ &= x\frac{[1-(1+i)^{-5}]}{(1+i)-1} \\ &= x\left[\frac{1-(1+i)^{-5}}{i}\right] \end{aligned}$$

So for the five payments made:

$$P_V = x\left[\frac{1-(1+i)^{-5}}{i}\right]$$

In general

If the first of *n* payments on a loan starts at the end of the first period then the standard timeline will be:



Notice that there is one period at the beginning of the timeline where nothing happens as the first payment is made at the end of the first period. (Ordinary annuity)

And the formula that applies will be $P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

Notice that the powers are negative because we are moving backward on the timeline towards the present value.

Example

Example 1

A loan of R100 000 is repaid by means of 10 semi-annual payments of Rx each. If interest on the loan is charged at 16% per annum compounded semi-annually,

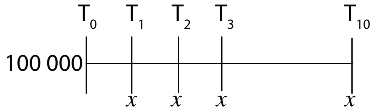
1. determine x if the first payment is made at the end of the first half year.
2. determine the semi-annual payment if the fist payment is in six months time and if a deposit of R15 000 was given.

Solution

1. First payment is made at the end of the first semi annum:

   We need an effective semi annual rate, so

   $i = \frac{0,16}{2}$ or $\frac{16}{200} = 0,08$; $P_V = 100\ 000$; $n = 10$



   $P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

   $\therefore\ 100\ 000 = x\left[\frac{1-(1,08)^{-10}}{0,08}\right]$

   $\therefore\ 100\ 000 = x(6,710081399)$

   $\therefore\ x = \text{R}14\ 902,95$

2. If we give a deposit, this money is taken off the value of the loan. So the loan amount will now be R100 000 – R15 000 = R85 000.

   So:

   $P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

   $\therefore\ 85\ 000 = x\left[\frac{1-(1,08)^{-10}}{0,08}\right]$

   $\therefore\ 85\ 000 = x(6,710081399)$

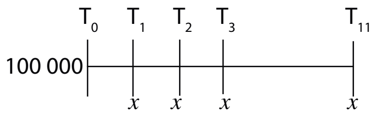
   $\therefore\ x = \text{R}12\ 667,51$

   It is very important to understand how the formula adjusts for different scenarios. Once you grasp this, you will not have a problem with the present value annuities.

Example 2

A loan of R1 000 is paid off by equal monthly payments of R88,85 per month at a rate of 12% p.a. compounded monthly. How long does it take to amortise the loan, if the first payment is made at the end of the first period.

This is an ordinary annuity, with timeline:



Notice that the period of this loan is unknown and that we will have to solve for *n*. Furthermore we also need to use an effective monthly rate so we have:

$i = \frac{0,12}{12}$ or $\frac{12}{1\ 200} = 0,01$

Now the formula applies:

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore\ 1\ 000 = 88,85\left[\frac{1-(1,01)^{-n}}{0,01}\right]$

$\therefore\ \left(\frac{1\ 000}{88,85}\right) \times 0,01 = 1-(1,01)^{-n}$

$\therefore\ (1,01)^{-n} = 1-\left(\frac{1\ 000}{88,85}\right) \times 0,01$

$\therefore\ (1,01)^{-n} = 0,8874507597$

$\therefore\ -n = \frac{\log 0,8874507597}{\log 1,01}$ or $\log_{1,01} 0,8874507597$

$\therefore = -11{,}999826$

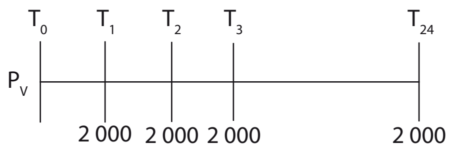
$\therefore n = 12$ months

Example

Example 3

A loan is amortised by 24 monthly payments of R2 000 each made into an ordinary annuity with interest charged at 14% per annum compounded monthly. Determine the value of the loan.

Notice that we do not have the loan amount, but we do have the future value of each of the payments. We need to remove the interest from these payments by bringing all 24 back to T_0:



The effective monthly rate will be:

$i = \frac{0{,}14}{12}$ or $\frac{14}{1\,200} = 0{,}011\dot{6}$

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore P_V = 2\,000\left[\frac{1-(\frac{14}{1200})^{-24}}{\frac{14}{1200}}\right]$

$\therefore P_V = 2\,000(20{,}82774314)$

$\therefore P_V = \text{R}41\,655{,}49$

If $i = 0{,}019$: RHS $= \frac{1-(1{,}019)^{-12}}{0{,}019} = 10{,}6405815 \neq$ LHS (We are moving further away)

Activity

Activity 3

1. A man plans to buy a house on a 24 year mortgage and can only afford to pay R2 700 per month. If the interest rate is currently 22% per annum compounded monthly, determine the size of the mortgage he can take, if he starts paying one month after the mortgage was approved.

2. A loan of R150 000 is amortized by equal quarterly payments for a period of 10 years at a rate of 14% p.a. compounded quarterly. Determine the size of each quarterly payment.

3. Louis buys a car at a purchase price of R75 000. He can repay this car in 24 equal monthly instalments, but has a choice of

3.1 14% p.a. simple interest

3.2 16% p.a. compounded monthly.

3.3 Which one should he choose and why?

4. Determine the amount that must be invested now to realise equal monthly withdrawals of R3 500 for the next 15 years if interest is at 16% p.a. compounded monthly. The first withdrawal will be in one months' time.

5. Sandra buys a slimming machine, and pays a deposit of R2 000 on the purchase. The balance is paid off by 36 equal monthly instalments of R 1800 each. Interest is calculated at 23% p.a. compounded monthly.

5.1 Calculate the purchase price of the slimming machine if the first payment is made at the end of the first month.

5.2 What amount would she have saved if she made the purchase cash?

5.3 If she took a loan for the balance and paid this loan back at 23% p.a. compounded quarterly with equal quarterly payments, would she have saved on the loan repayments?

6. A competition makes a startling claim that you can win a prize of 1 million rand. The small print informs us that the prize will be paid out in equal annual instalments of R50 000 over the next 20 years, starting now. Assuming an average inflation rate of 14% p.a. over the next twenty years, show that the present value of the prize is significantly less than the claimed 1 million rand.

7. When considering the purchase of a house, Mr Pillay has to take the following into account :

- The house is on the market for R570 000
- He has R80 000 available as a deposit
- The Bank's condition for granting a mortgage bond (loan) for the balance is that his monthly repayments may not exceed $\frac{1}{3}$ of his monthly salary.
- His salary is R17 500 per month
- The bank is offering mortgage bonds at 16,25% p.a compounded monthly, repayable in equal monthly installments over 20 years.

7.1 Show that Mr Pillay does not meet the 'third requirement' above. Set your argument out clearly.

7.2 As he is determined to purchase the house, he decides on a two-pronged strategy :

- to put in an offer to purchase which is R50 000 less than the asking price;
- to ask the bank to let him repay the bond over a longer period

Calculate how many years he will need to pay off the loan.

Outstanding balance on a loan

Often a person with debt decides to settle their debts when they come across some money by winning a lotto game or by inheriting some money from a friend or a relative. We also hear quite often that the governor of the reserve bank announces a change in the interest rate. In both these scenarios the banks have to find an outstanding balance on your loan, to calculate what amount of the original capital amount of the loan, is still owed to the bank.

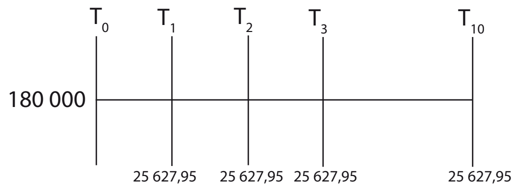
Let's see how the capital amount gets smaller with each payment:

A loan of R180 000 is repaid by means of 10 semi-annual payments of R x each. Interest on the loan is charged at 14% per annum compounded semi-annually.

If the first payment was made at the end of the first period (an ordinary annuity) the semi-annual payment will be R25 627,95:

The effective semi anuual rate will be
$i = \frac{0{,}14}{2}$ or $\frac{14}{200} = 0{,}07$

The timeline:



$$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$$
$$\therefore 180\,000 = x\left[\frac{1-(1{,}07)^{-10}}{0{,}07}\right]$$
$$\therefore x = \frac{180\,000 \times 0{,}07}{1-(1{,}07)^{-10}}$$
$$\therefore x = \text{R}25\,627{,}95$$

Let us discuss the outcomes of the example above:

For the ordinary annuity, the first payment is made at the end of six months (semi-annum). Consider the table below:

Timeline	Balance at Timeline	Balance Outstanding after payment	Regular Payments	Interest
0	180000,00			
1	192600,00	166972,05	25627,95	12600,00
2	178660,09	153032,14	25627,95	11688,04
3	163744,39	138116,44	25627,95	10712,25
4	147784,59	122156,64	25627,95	9668,15
5	130707,61	105079,66	25627,95	8550,96
6	112435,24	86807,29	25627,95	7355,58
7	92883,80	67255,85	25627,95	6076,51
8	71963,76	46335,81	25627,95	4707,91
9	49579,31	23951,36	25627,95	3243,51
10	25627,95	0	25627,95	1676,59
10 payments		Owing R0	Total paid: R256279,50	Total paid in interest: R76279,50

If we want to manually calculate the portion of interest on the first payment we need to do the following:

$\underbrace{180\,000(1{,}07)}_{\text{the first semi annum interest is added to the value of the loan}} - 180\,000(0{,}07) = 180\,000(0{,}07) = \text{R}12\,600$

So we could have said R180 000 × 0,07 = R12 600. To then calculate the capital amount paid we take the regular payment and subtract the interest:

R25 627,95 – R12 600 = R13 027,95.

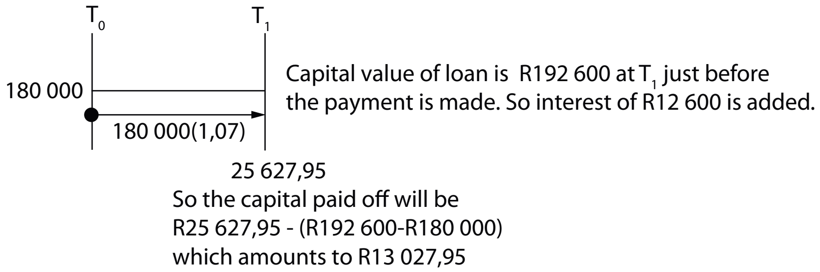
LIBERTY LIFE

The timeline below illustrates this clearly.

If we wish to calculate the balance outstanding after the fifth payment, we can do this in one of two ways. These methods are based on the fact that you **must move to the same point on the timeline** to be able to compare things.

Method 1:

Our first method moves forward on the timeline as is shown in the diagram below.



We need to move the **original amount** of the loan forward with interest, and also the **regular payments** that were made.

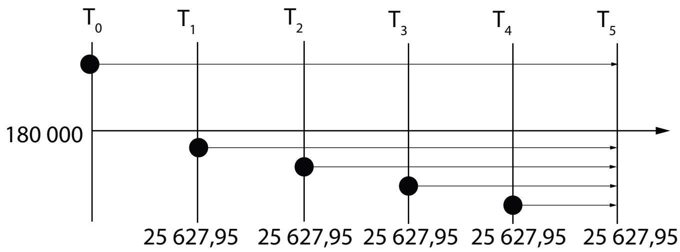
Balance of the loan just after the payment is made at T5:

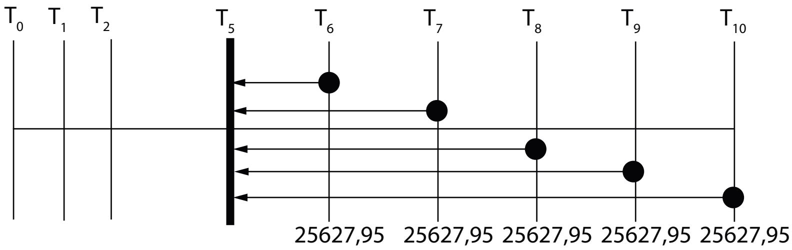
(loan + interest) – (instalments + interest)

$180\,000(1{,}07)^5 - 25\,627{,}95\left[\frac{(1{,}07)^5 - 1}{0{,}07}\right] = 105\,079{,}66.$

Method 2:

This method just focuses on the payments that still have to be made and works backwards on the timeline by calculating the present value of these payments at T5.





We need to move each payment that has not been made back to timeline T5. This would mean removing interest from each payment up to T5.

So we are thus working with the present value of an annuity:

$OB_{T_5} = 25\,627{,}95\left[\frac{1 - (1{,}07)^{-5}}{0{,}07}\right]$

$= 25\,627{,}95(4{,}100197436)$

$= R105\,079{,}66$

It is clear that method 2 will be more time efficient for you to apply.

At this point it is important to draw your attention to some important facts about outstanding balance:

- Outstanding balance on any loan is always calculated directly after the last payment is made.
- Method two suggests that the outstanding balance is the **present value of all payments yet to be made.**

Example

Example 1

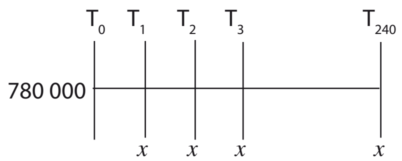
Lets solve some problems where the outstanding balance is needed.

Trevor purchases a house for R780 000 on a mortgage for 20 years with interest at 17% p.a. compounded monthly. The mortgage payments are made at the end of each month.

1. Calculate the monthly instalment
2. Calculate the outstanding balance after 10 years
3. At this point, how much has been paid into the bond account and how much capital was paid off on this loan?
4. Calculate the new monthly payment if the interest rate changes to 19% p.a. compounded monthly after ten years.
5. If Trevor rather asked the bank for a 25 year mortgage, assuming that the rate remains fixed at 17% p.a. compounded monthly for the duration of the loan, how much less will he pay every month?

Solution

1.



The effective monthly rate is:

$$i = \frac{0{,}17}{12} \text{ or } \frac{17}{1200} = 0{,}014166666\ldots = 0{,}0141\dot{6}$$

So:

$$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\therefore\ 780\,000 = x\left[\frac{1-(1{,}0141\dot{6})^{-240}}{0{,}0141\dot{6}}\right]$$

$$\therefore\ 780\,000 = x(68{,}17559487)$$

$$\therefore\ x = \text{R}11\,441{,}04$$

2. After 10 years, the number of payments that were made is 120. There is thus still 120 payments left. So:

$$OB_{T_{120}} = 11\,441{,}04\left[\frac{1-(1{,}0141\dot{6})^{-120}}{0{,}0141\dot{6}}\right]$$

$$= 11\,441{,}04(57{,}53817667)$$

$$= \text{R}658\,296{,}58$$

3. After ten years, Trevor has paid into the bond account:

$$120 \times \text{R}11\,441{,}04 = \text{R}1\,372\,924{,}80$$

The capital paid off on this loan after paying for ten years:

R780 000 – R658 296,58

$= R121\ 703,42$.

4. The new effective monthly rate will be:

$i = \frac{0,19}{12}$ or $\frac{19}{1200} = 0,015833333\ldots = 0,0158\dot{3}$

The outstanding balance at this point is given as R658 296,58, so:

$658\ 296,58 = x\left[\frac{1-(1,0158\dot{3})^{-120}}{0,0158\dot{3}}\right]$

$\therefore\ 658\ 296,58 = x(53,56979602)$

$\therefore\ x = R12\ 288,58$

So Trevor will now have to pay R12 288,58 on his bond with the interest rate increase.

5. Asking the bank for a 25 year mortgage might save money monthly, but in the long run you pay more for the house:

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore\ 780\ 000 = x\left[\frac{1-(1,0141\dot{6})^{-300}}{0,0141\dot{6}}\right]$

$\therefore\ 780\ 000 = (69,55086772)$

$\therefore\ x = R11\ 214,81$

He will only save R226,23 per month.

Activity 4

Activity

1. Mark wants to purchase a big screen television set at a cash price of R 39 000. He wishes to obtain a loan from the bank at an interest rate of 18% p.a. compounded monthly. Mark can only afford to pay R1 200 per month for a period of 36 months.

1.1 Will the bank give him a loan based on this information?

1.2 If the bank agrees to accept R1 200 per month, how long will it take Mark to pay back the loan?

2. A loan of R120 000 is repaid by 72 equal monthly payments into an ordinary annuity of Rx each. The interest rate charged is 16% per annum compounded monthly.

2.1 Determine the amount of the monthly payment.

2.2 Determine the balance outstanding after the 24th payment.

2.3 If the interest rate is changed to 18% per annum compounded monthly, directly after the 24^{th} payment, determine the new monthly payment to settle the loan over the same time period.

LIBERTY LIFE

3. Brad purchased a car for R72 000. He agrees to a loan of 54 months at a rate of 19,25% p.a. compounded monthly, with his first payment due at the end of the first month. After the 36^{th} payment, the bank agrees that he can settle the account. How much must he pay the bank?

4. A house that was bought 8 years ago for R50 000 is now worth R100 000. Originally the house was financed by paying 20% deposit with the rest financed through a 20 year mortgage at 10,5% interest per annum compounded monthly. The owner, after making 96 equal monthly payments, is in need of cash, and would like to refinance the house. The finance company is willing to loan 80% of the new value of the house, less any amount still owing. How much cash will the owner be able to borrow?

5. Mathew bought a house for R825 000 in Wapadrand and took a mortgage bond at a rate of 16,25% p.a. compounded monthly. This bond has to be paid back over a period of 20 years by making monthly deposits into the mortgage account. If Mathew put down a deposit of R84 000,

5.1 Determine the monthly payment on his bond account.

5.2 After the 200th payment, the bank rate changes to 17,5% p.a. compounded monthly. Determine the outstanding balance on the loan at this time

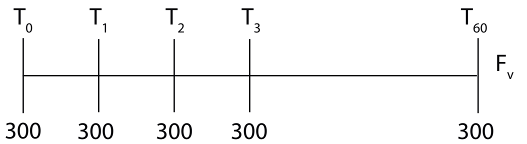
5.3 Determine the new monthly payment that will amortize the loan.

5.4 How much did Mathew actually pay for his house?

Solutions to Activities

Activity 1

1.



The effective monthly rate is $\frac{0,14}{12}$ or $\frac{14}{1\,200} = 0,011\dot{6}$

Notice that the payments start immediately and end on the last day, so 61 payments are made.

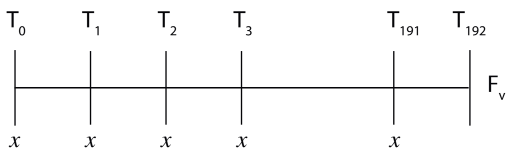
$F_V = x\left(\frac{(1+i)^n - 1}{i}\right)$

$\therefore\ F_V = 300\left(\frac{(1,011666\ldots)^{61} - 1}{0,01166666\ldots}\right)$

$\therefore\ F_V = 300(88,20073489)$

$\therefore\ F_V = \text{R}26\,460,22$

2.1



Effective monthly rate: $\frac{0,08}{12}$ or $\frac{8}{1\,200} = 0,00\dot{6}$

Note this is an annuity due with $16 \times 12 = 192$ payments made into the annuity.

$F_V = x\left(\frac{(1+i)^n - 1}{i}\right)(1+i)$

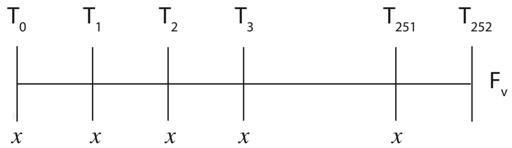
$\therefore\ 16\,000 = x\left(\frac{(1,00666\ldots)^{192} - 1}{0,00666\ldots}\right)(1,00666\ldots)$

$\therefore\ 16\,000 = x(387,2091494)(1,00666\ldots)$

$\therefore$ 16 000 = x(389,7905437)

$\therefore$ x = R41,05

2.2



Effective monthly rate: $\frac{0,08}{12}$ or $\frac{8}{1\,200} = 0,00\dot{6}$

Note that the period is changing to accommodate 21 × 12=252 payments

$F_V = x\left(\frac{(1+i)^n - 1}{i}\right)(1+i)$

$\therefore 21\,000 = x\left(\frac{(1,00666\ldots)^{252} - 1}{0,00666\ldots}\right)(1,00666\ldots)$

$\therefore 21\,000 = x(650,3587456)(1,00666\ldots)$

$\therefore 21\,000 = x(654,6944706)$

$\therefore x = \text{R}32,08$

3. For 12% p.a. c.m. – effective $\frac{0,12}{12}$ or $\frac{12}{1\,200} = 0,01$:

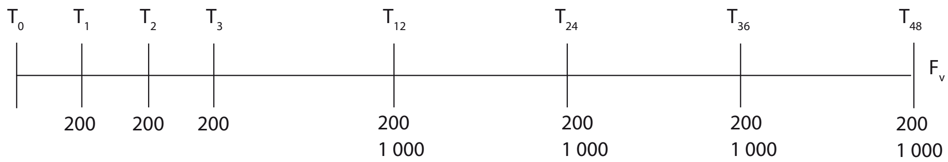
$100\left(\frac{(1,01)^{12} - 1}{0,01}\right) = 1\,268,25$

For 4% p.q. c.q. – this is an effective rate so here $i = 0,04$:

$300\left(\frac{(1,04)^4 - 1}{0,04}\right) = 1\,273,94$

So clearly the quarterly investment is better at these rates that were given.

4.



Effective monthly rate is needed for the monthly deposits:

$\frac{0,06}{12}$ or $\frac{6}{1\,200} = 0,005$

The effective annual rate is needed for the annual deposits:

$(1+i) = \left(1 + \frac{i_{12}}{12}\right)^{12}$

$\therefore i = \left(1 + \frac{0,06}{12}\right)^{12} - 1$

$\therefore i = 0,06167781186$

We treat this scenario as two separate annuities that are running at the same time:

$F_V = 200\left(\frac{(1,005)^{48} - 1}{0,005}\right) + 1\,000\left(\frac{(1,06167781186)^4 - 1}{0,06167781186}\right)$

the first annuity at R200p.m. the second annuity at R1 000 p.a.

$\therefore F_V = 200(54,09783222) + 1\,000(4,385518113)$

$\therefore F_V = \text{R}15\,205,08$

Activity 2

1. Scrap Value = 0,18x

Depreciation:

$F_V = P_V(1 - i)^n$

$\therefore 0,18x = x(1 - 0,21)^n$

LIBERTY LIFE

$\therefore 0{,}18 = (0{,}79)^n$

$\therefore n = \frac{\log 0{,}18}{\log 0{,}79}$

$\therefore n = 7{,}274654049$

$\therefore n = 7{,}27$ years

2.1 $F_V = P_V(1 - i)^n$

$\therefore 45\,000 = 120\,000(1 - i)^{10}$

$\therefore 0{,}375 = (1 - i)^{10}$

$\therefore 1 - i = \sqrt[10]{0{,}375}$

$\therefore -i = -0{,}09342627726$

$\therefore i = 0{,}0934$

$\therefore r = 9{,}34\%$ p.a.c.a

2.2 $F_V = P_V(1 - i{\cdot}n)$

$\therefore 45\,000 = 120\,000(1 - 10i)$

$\therefore 0{,}375 = (1 - 10i)$

$\therefore 10i = 1 - 0{,}375$

$\therefore 10i = 0{,}625$

$\therefore r = 6{,}25\% SI$

3.1 $F_V = P_V(1 - i)^n$

$\therefore F_V = 550\,000(1 - 0{,}1)^8$

$\therefore F_V = R236\,756{,}97$

3.2 $F_V = P_V\,(1 + i)n$

$\therefore F_V = 550\,000(1{,}18)^8$

$\therefore F_V = R2\,067\,372{,}56$

3.3 R2 067 372,56 – R236 756,97 = R 1 830 615,59.

3.4 $F_V = x\left(\frac{(1 + i)^n - 1}{i}\right)$

$\therefore 1\,830\,615{,}59 = x\left(\frac{(1{,}01)^{97} - 1}{0{,}01}\right)$

$\therefore 1\,830\,615{,}59 = x(162{,}5265655)$

$\therefore F_V = R11\,263{,}49$

4.1 $F_V = P_V(1 - i{\cdot}n)$

$\therefore F_V = 850\,000(1 - 0{,}07 \times 8)$

$\therefore F_V = 850\,000(0{,}44)$

$\therefore F_v = R374\,000$

4.2 $F_V = P_V(1 + i)^n$

$\therefore F_V = 850\,000(1{,}14)^8$

$\therefore F_V = R2\,424\,698{,}46$

4.3 Sinking fund = R 2 050 698,46

4.4 $F_V = x\left(\frac{(1{,}0125)^{90} - 1}{0{,}0125}\right)(1{,}0125)^6$

$\therefore 2\,050\,698{,}46 = x\left(\frac{(1{,}0125)^{90} - 1}{0{,}0125}\right)(1{,}0125)^6$

$\therefore 2\ 050\ 698{,}46 = x(164{,}7050076)(1{,}0125)^6$

$\therefore x = \text{R}11\ 556{,}46$

5.1 $F_V = P_V(1 - i)^n$

$\therefore F_V = 1\ 200\ 000(1 - 0{,}09)^5$

$F_V = \text{R}748\ 838{,}57$

5.2 $F_V = P_V(1 + i)^n$

$\therefore F_V = 1\ 200\ 000(1{,}06)^5$

$\therefore F_V = \text{R}1\ 605\ 870{,}69$

5.3 Sinking fund = R 857 032,12

$F_V = x\left(\frac{(1+i)^n - 1}{i}\right)$

$\therefore\ 857\ 032{,}12 = x\left(\frac{(1{,}035)^{21} - 1}{0{,}035}\right)$

$\therefore\ 857\ 032{,}12 = x(30{,}26947068)$

$\therefore x = \text{R}28\ 313{,}42$

5.4 **Scrap Value of current equipment:**

$F_V = P_V(1 - i)^n$

$\therefore F_V = 1200\ 000(1 - 0{,}09)^3$

$\therefore F_V = \text{R}904\ 285{,}20$

In the sinking fund:

$F_V = x\left(\frac{(1+i)^n - 1}{i}\right)$

$\therefore F_V = 28\ 313{,}42\left(\frac{(1{,}035)^{13} - 1}{0{,}035}\right)$

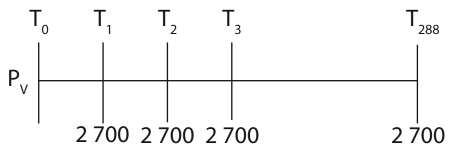
$\therefore F_V = 28\ 313{,}42(16{,}1130303)$

$\therefore F_V = \text{R}456\ 214{,}99$

She will have R1 360 500,19 available.

Activity 3

1.



the effective monthly rate will be:

$i = \frac{0{,}22}{12}$ or $\frac{22}{1\ 200} = 0{,}018\dot{3}$

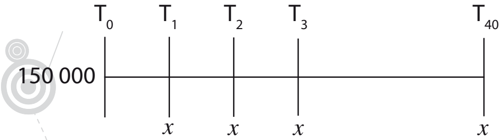
$P_V = x\left[\frac{1 - (1+i)^{-n}}{i}\right]$

$\therefore\ P_V = 2\ 700\left[\frac{1 - (1{,}018333333)^{-288}}{0{,}0183333333}\right]$

$\therefore\ P_V = 2\ 700(54{,}254081)$

$\therefore\ P_V = \text{R}146\ 486{,}02$

2.



- the effective quarterly rate will be:

$i = \frac{0{,}14}{4}$ or $\frac{14}{400} = 0{,}035$

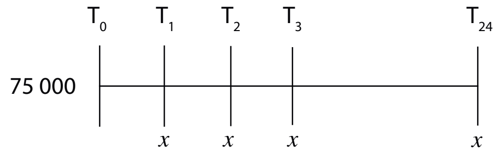
$P_V = x\left[\frac{1 - (1+i)^{-n}}{i}\right]$

$\therefore\ 150\ 000 = x\left[\frac{1 - (1{,}035)^{-40}}{0{,}035}\right]$

$\therefore\ 150\ 000 = x(21{,}35507234)$

$\therefore\ P_V = \text{R}7\ 024{,}09$

3.



3.1 The rate is a simple interest rate, so this is a hire purchase agreement:

Now:

$x = \frac{F_V}{24} = \frac{P_V(1 + i.n)}{24}$

$\therefore x = \frac{75\ 000(1 + 0{,}14 \times 2)}{24}$

$\therefore x = \text{R}4\ 000$

So the instalment is R4 000 per month at a simple interest rate.

3.2 the effective monthly rate will be:

$i = \frac{0{,}16}{12}$ or $\frac{16}{1\ 200} = 0{,}01\dot{3}$

$P_V = x\left[\frac{1 - (1 + i)^{-n}}{i}\right]$

$\therefore 75\ 000 = x\left[\frac{1 - (1{,}01\dot{3})^{-24}}{0{,}01\dot{3}}\right]$

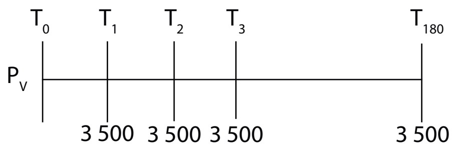
$\therefore 75\ 000 = x(20{,}42353906)$

$\therefore x = \text{R}3\ 672{,}23$

3.3 Louis should opt for the compound interest agreement as he will save:

$24 \times (4\ 000 - 3\ 672{,}23) =$ R7 866,48 on the deal.

4.



$P_V = x\left[\frac{1 - (1 + i)^{-n}}{i}\right]$

$\therefore P_V = 3\ 500\left[\frac{1 - (1{,}01333333)^{-180}}{0{,}013333333}\right]$

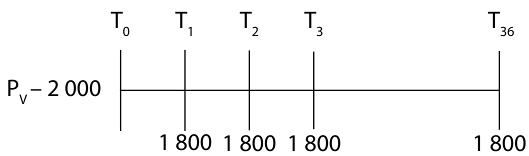
$\therefore P_V = 3\ 500(68{,}08738987)$

$\therefore P_V = \text{R}238\ 305{,}86$

The effective monthly rate:

$i = \frac{0{,}16}{12}$ or $\frac{16}{1\ 200} = 0{,}01\dot{3}$

5.



The effective monthly rate:

$i = \frac{0{,}23}{12}$ or $\frac{23}{1\ 200} = 0{,}0191\dot{6}$

5.1 $P_V - 2\ 000 = x\left[\frac{1 - (1 + i)^{-n}}{i}\right]$

$\therefore P_V - 2\ 000 = 1\ 800\left[\frac{1 - (1{,}0191666\ldots)^{-36}}{0{,}0191666\ldots}\right]$

$\therefore P_V - 2\ 000 = 1\ 800(25{,}83330388)$

$\therefore P_V = \text{R}46\ 499{,}95 + \text{R}2\ 000$

$\therefore P_V = \text{R}48\ 499{,}95$

5.2 She paid $36 \times 1\ 800 + 2\ 000 = \text{R}66\ 800$

If she purchased it cash she would have paid R48 499,95. This means she would have saved R18 300,05.

5.3 The effective quarterly rate:

$i = \frac{0{,}23}{4}$ or $\frac{23}{400} = 0{,}0575$

The period will remain 3 years but the number of payments will be 12 quarterly payments.

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore\ 46\,499{,}95 = x\left[\frac{1-(1{,}0575)^{-12}}{0{,}0575}\right]$

$\therefore\ 46\,499{,}95 = x(8{,}499955647)$

$\therefore\ x = \text{R}5\,470{,}61$

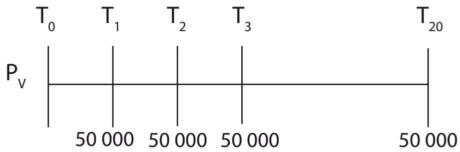
She now pays

$12 \times \text{R}5\,470{,}61 + \text{R}2\,000 = \text{R}67\,647{,}32$.

So she would not have saved – in fact she pays R847,32 more if the payments happen quarterly.

6. The effective yearly rate will be: $i = \frac{14}{100} = 0{,}14$.

The timeline:



$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore\ P_V = 50\,000\left[\frac{1-(1{,}14)^{-20}}{0{,}14}\right]$

$\therefore\ P_V = 50\,000(6{,}623130552)$

$\therefore\ P_V = \text{R}331\,156{,}53$

This is R668 843,47 short of R1 million

7.1 He gives a deposit of R80 000. So the loan amount will be R490 000. His monthly salary is R17 500 and a third of this is R5 833,33.

The effective monthly rate is:

$\text{i} = \frac{0{,}1625}{12}$ or $\frac{16{,}25}{1\,200} = 0{,}013541\dot{6}$

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore\ 490\,000 = x\left[\frac{1-(1{,}013541\dot{6})^{-240}}{0{,}013541\dot{6}}\right]$

$\therefore\ 490\,000 = x(70{,}91969686)$

$\therefore\ x = \text{R}6\,909{,}22$

Mr Pillay does not qualify by R1 075,89.

7.2 If he pays a deposit of R80 000 and offers R50 000 less, the bond amount will be R440 000. He can only afford to pay R5 833,33 per month. So:

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore\ 440\,000 = 5\,833{,}33\left[\frac{1-(1{,}013541\dot{6})^{-n}}{0{,}013541\dot{6}}\right]$

$\therefore\ \frac{440\,000}{5\,833{,}33} \times 0{,}01354166\ldots = 1-(1{,}013541\dot{6})^{-n}$

$\therefore\ (1{,}013541\dot{6})^{-n} = 0{,}021429155$

$\therefore\ -n = \frac{\log 0{,}021429155}{\log 1{,}013541\dot{6}}$

$\therefore\ n = 285{,}7081773$ (this answer is in months, so ÷ by 12)

$\therefore\ n = 23$ years and 10 months

Activity 4

1.1 We need to find the present value of the 36 payments of R1500 each. The effective monthly rate will be: $i = \frac{0{,}18}{12}$ or $\frac{18}{1200} = 0{,}015$.

Now:

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore P_V = 1\,200\left[\frac{1-(1{,}015)^{-36}}{0{,}015}\right]$

$\therefore P_V = 1\,200(27{,}66068431)$

$\therefore x = \text{R}33\,192{,}82$

Paying R1200 per month for 36 months does not settle the loan which is R39000. He still shorts R5807,18.

1.2 If the bank accepts the R1200 per month, then the period will be longer than 36 months. We need to find n for the present value of R39000:

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore 39\,000 = 1\,200\left[\frac{1-(1{,}015)^{-n}}{0{,}015}\right]$

$\therefore \frac{39\,000}{1\,200} \times 0{,}015 = 1-(1{,}015)^{-n}$

$\therefore (1{,}015)^{-n} = 1 - \frac{39000}{1200} \times 0{,}015$

$\therefore (1{,}015)^{-n} = 0{,}5125$

$\therefore -n = \frac{\log 0{,}5125}{\log 1{,}015}$ or $\log_{1{,}015}(0{,}5125) = -44{,}8970\ldots$

$\therefore n = 45$ months

2.1 The effective monthly rate will be:

$i = \frac{0{,}16}{12}$ or $\frac{16}{1\,200} = 0{,}01\dot{3}$.

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore 120\,000 = x\left[\frac{1-(1{,}01\dot{3})^{-72}}{0{,}01\dot{3}}\right]$

$\therefore 120\,000 = x(46{,}10028344)$

$\therefore x = \text{R}2\,603{,}02$

2.2 After the 24th payment, there are still 48 payments left to be made.

SO:

$OB_{T_{24}} = 2\,603{,}02\left[\frac{1-(1{,}01\dot{3})^{-48}}{0{,}01\dot{3}}\right]$

$= 2\,603{,}02(35{,}28546548)$

$= \text{R}91\,848{,}77$

2.3 For the balance of R91 848,77 the new effective monthly rate will be:

$i = \frac{0{,}18}{12}$ or $\frac{18}{1\,200} = 0{,}015$

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore\ 91\,848{,}77 = x\left[\frac{1-(1{,}015)^{-48}}{0{,}015}\right]$

$\therefore\ 91\,848{,}77 = x(34{,}04255365)$

$\therefore\ x = \text{R}2\,698{,}06$

3. We first need to find how much Brad must pay the bank monthly.

The effective monthly rate is $i = \frac{0{,}1925}{12}$ or $\frac{19{,}25}{1\,200} = 0{,}016041\dot{6}$.

$$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\therefore\ 72\ 000 = x\left[\frac{1-(1{,}016041\dot{6})^{-54}}{0{,}016041\dot{6}}\right]$$

$$\therefore\ 72\ 000 = x(35{,}94224527)$$

$$\therefore\ x = \text{R}2\ 003{,}21$$

To settle after the 36th payment, we obtain the present value of the last 18 payments on this day:

$$OB_{36} = x\left[\frac{1-(1+i)^{-18}}{i}\right]$$

$$= 2\ 003{,}21\left[\frac{1-(1{,}016041\dot{6})^{-18}}{0{,}016041\dot{6}}\right]$$

$$= 2\ 003{,}21(15{,}52717265)$$

$$= \text{R}31\ 104{,}19$$

He thus has to pay R31 104,19.

4. We firstly need to work out what the monthly payments were on the original loan of

R50 000 – deposit:

Loan amount = R50 000 – 20% of R50 000

= R50 000 – R10 000

= R40 000

The interest rate: $i = \frac{0{,}105}{12}$ or $\frac{10{,}5}{1\ 200} = 0{,}00875$.

So:

$$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\therefore 40\ 000 = x\left[\frac{1-(1{,}00875)^{-240}}{0{,}00875}\right]$$

$$\therefore 40\ 000 = x(100{,}1622742)$$

$$\therefore x = \text{R}399{,}35$$

After 96 months, the house is worth R100 000, and the bank will loan R80000 to the homeowner. We now need the outstanding balance after 8 years – that is the present value of the last 144 payments that must still be made:

$$\text{OB}_{96} = x\left[\frac{1-(1+i)^{-144}}{i}\right]$$

$$= 399{,}35\left[\frac{1-(1{,}00875)^{-144}}{0{,}00875}\right]$$

$$= 399{,}35(81{,}68995711)$$

$$= \text{R}32\ 622{,}88$$

So he will receive:

R80 000 – R32 622,88 = R47 377,12.

5.1 Effective monthly rate:

$$i = \frac{0{,}1625}{12} \text{ or } \frac{16{,}25}{1200} = 0{,}013541\dot{6}$$

$$P_V - \text{deposit} = x\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\therefore 825\ 000 - 84\ 000 = x\left[\frac{1-(1{,}013541\dot{6})^{-240}}{0{,}013541\dot{6}}\right]$$

$$\therefore 741\ 000 = x(70{,}91969686)$$

$\therefore x = \text{R}10\,448{,}44$

5.2 There are 40 payments left:

$OB_{200} = 10\,448{,}44 \left[\frac{1-(1{,}013541\dot{6})^{-40}}{0{,}013541\dot{6}}\right]$

$= 10\,448{,}44(30{,}72765908)$

$= \text{R}321\,056{,}10$

5.3 The new monthly payment for the remaining R321 056,10 at an effective rate of $i = \frac{0{,}175}{12}$ or $\frac{17{,}5}{1\,200} = 0{,}01458\dot{3}$:

$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

$\therefore 321\,056{,}10 = x\left[\frac{1-(1{,}01458\dot{3})^{-40}}{0{,}01458\dot{3}}\right]$

$\therefore 321\,056{,}10 = x(30{,}14462647)$

$\therefore x = \text{R}10\,650{,}53$

5.4 Mathew paid a total of:

$200 \times \text{R}10\,448{,}44 + 40 \times \text{R}10\,650{,}53 + \text{R}84\,000$

$= \text{R}2\,599\,709{,}20$ for his house.